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Bridge Graphs and Deodhar Parametrizations for Positroid Varieties

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Abstract. A *parametrization* of a positroid variety Π of dimension d is a regular map $(\mathbb{C}^{\times})^{d} \to \Pi$ which is birational onto a dense subset of Π. There are several remarkable combinatorial constructions which yield parametrizations of positroid varieties. We investigate the relationship between two families of such parametrizations, and prove they are essentially the same. Our first family is defined in terms of Postnikov's *boundary measurement map*, and the domain of each parametrization is the space of edge weights of a planar network. We focus on a special class of planar networks called *bridge graphs*, which have applications to particle physics. Our second family arises from Marsh and Rietsch's parametrizations of Deodhar components of the flag variety, which are indexed by certain subexpressions of reduced words. Projecting to the Grassmannian gives a family of parametrizations for each positroid variety. We show that each Deodhar parametrization for a positroid variety corresponds to a bridge graph, while each parametrization from a bridge graph agrees with some projected Deodhar parametrization.

Résumé. Soit Π une variété positroïde. Nous appellerons *paramétrisation* toute application régulière $(\mathbb{C}^{\times d}) \to \Pi$ qui est un isomorphisme birégulier sur un sous-ensemble dense de Π. On sait que plusieurs constructions combinatoires donnent des paramétrisations intéressantes. Le but du présent article est d'investiguer deux familles de telles paramétrisations et de montrer, essentiellement, qu'elles coïncident. La première famille trouve son origine dans la *fonction de mesure des bords* de Postnikov. Le domaine de chaque paramétrisation est en ce cas-ci l'ensemble de poids des arêtes d'un réseau planaire pondéré. Nous nous concentrons sur une classe particulière de réseaux planaires, les *graphes de ponts*, ayant des applications à la physique subatomique. La deuxième famille provient des paramétrisations de Marsh et de Rietsch des composantes de Deodhar (indexées par certaines sous-expressions de mots réduits de permutations) de la variété de drapeaux. On obtient alors des paramétrisations de cellules de positroïdes en appliquant la projection à la grassmannienne. Nous montrons que chaque paramétrisation de Deodhar corresponde à un graphe de ponts; d'autre part, chaque paramétrisation provenant d'un graphe de ponts s'accorde avec quelque paramétrisation de Deodhar.

Keywords: positroids varieties, plabic graphs, bridge graphs, bounded affine permutations, Deodhar parametrizations, positive distinguished subexpressions

1 Introduction

Lusztig defined the *totally nonnegative part* of an abstract flag manifold and conjectured that it was made up of cells, a conjecture later proved by Rietsch (Lusztig, 1994, 1998; Rietsch, 1999). More than a

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decade later, Postnikov introduced the *positroid stratification* of the totally nonnegative Grassmannian $\mathrm{Gr}_{\geq 0}(k,n)$, which Rietsch then showed was a special case of Lusztig's stratification (Postnikov, 2006; Rietsch, 2009). While Lusztig's approach relied on the machinery of canonical bases, Postnikov's was more elementary. Each of Postnikov's *positroid cells* is the locus in $\mathrm{Gr}_{\geq 0}(k,n)$ where certain Plücker coordinates vanish.

The positroid stratification of $\mathrm{Gr}_{\geq 0}(k,n)$ extends to a stratification of the complex Grassmannian $\mathrm{Gr}(k,n)$ of k-planes in n-space. That is, we can decompose $\mathrm{Gr}(k,n)$ into *open positroid varieties* $\mathring{\Pi}$ whose intersections with $\mathrm{Gr}_{\geq 0}(k,n)$ are precisely Postnikov's totally nonnegative cells $\mathring{\Pi}_{\geq 0}$. Remarkably, open positroid varieties are the images of *open Richardson varieties* in the flag variety $\mathcal{F}\ell(n)$ under the natural projection $\pi_k : \mathcal{F}\ell(n) \to \mathrm{Gr}(k,n)$.

The decomposition of $\mathrm{Gr}(k,n)$ into projected Richardson varieties was first studied by Lusztig (Lusztig, 1998). Rietsch showed that this decomposition is a stratification, and described the closure partial order on the cells (Rietsch, 2006). Brown, Goodearl and Yakimov investigated the same stratification from the viewpoint of Poisson geometry (Brown et al., 2006). Finally, Knutson, Lam and Speyer showed that Lusztig's (closed) strata were in fact the Zariski closures of Postnikov's totally nonnegative cells (Knutson et al., 2013).

Postnikov defined a family of maps onto each positroid cell in $\mathrm{Gr}_{\geq 0}(k,n)$. The domain of each map is the space of positive real edge weights of some weighted planar network, and there is a class of such networks for each positroid cell (Postnikov, 2006). If we let the edge weights range over $\mathbb{C}^\times$ instead of $\mathbb{R}^+$ we obtain a well-defined map onto a dense subset of the positroid variety $\mathring{\Pi}$ in $\mathrm{Gr}(k,n)$ corresponding to the totally nonnegative cell $\mathring{\Pi}_{\geq 0}$ (Muller and Speyer, 2014). Specializing all but an appropriately chosen set of edge weights to 1 gives a birational map to a dense subset of $\mathring{\Pi}$, which we call an *parametrization*. In this paper, we investigate a particular class of network parametrizations, which arise from *bridge graphs*. Bridge graphs are constructed by an inductive process, and the definition of the corresponding parametrization is particularly straightforward. In addition, bridge graphs have proven to be a useful tool in particle physics (Arkani-Hamed et al., 2012).

Our second family of parametrizations arises from Deodhar's decompositions of $\mathcal{F}\ell(n)$ (Deodhar, 1985). Each Deodhar decomposition of $\mathcal{F}\ell(n)$ refines the *Richardson decomposition*. Richardson varieties are indexed by intervals $[u,w]$ in Bruhat order on the symmetric group S_n. To define a Deodhar decomposition, choose a reduced word $\mathbf{w}$ for each $w \in S_n$. For each chosen $\mathbf{w}$, and each $u \leq w$ in Bruhat order, we can express the (open) Richardson variety $\mathring{X}_u^w$ indexed by $[u,w]$ as a disjoint union of *Deodhar components*. The Deodhar components of $\mathring{X}_u^w$ are indexed by *distinguished subexpressions* for u in $\mathbf{w}$; that is, by subwords for u in $\mathbf{w}$ which satisfy a technical condition.

Let $\mathring{\Pi} \subset \mathrm{Gr}(k,n)$ be a positroid variety, and fix a Deodhar decomposition of $\mathcal{F}\ell(n)$. We have a family of Deodhar components $\mathcal{D} \subset \mathcal{F}\ell(n)$ such that the natural projection from $\mathcal{F}\ell(n)$ to $\mathrm{Gr}(k,n)$ maps each $\mathcal{D}$ isomorphically to a dense subset of $\mathring{\Pi}$. These are precisely the top-dimensional Deodhar components of the Richardson varieties $\mathring{X}_u^w$ which project birationally to $\mathring{\Pi}$. For each such $\mathring{X}_u^w$, the desired component is indexed by a special choice of subexpression for u in $\mathbf{w}$ called a *positive distinguished subexpression*, or PDS. Marsh and Rietsch defined matrix parametrizations for each Deodhar component of $\mathcal{F}\ell(n)$ (Marsh and Rietsch, 2004). Composing with π_k gives a family of parametrizations for the positroid variety $\mathring{\Pi}$, which we call *projected Deodhar parametrizations* or simply *Deodhar parametrizations* (Talaska and Williams, 2013).

We will show that these two ways of parametrizing positroid varieties–via bridge graphs, and via projected Deodhar parametrizations–are essentially the same. This result was first conjectured by Thomas Lam (Lam, 2013a). Our main result is the following.

Theorem 1.1. *Let $\mathring{\Pi}$ be a positroid variety in $\mathrm{Gr}(k,n)$. For each Deodhar parametrization of $\mathring{\Pi}$, there is a bridge graph which yields the same parametrization. Conversely, any bridge graph parametrization of $\mathring{\Pi}$ agrees with some Deodhar parametrization.*

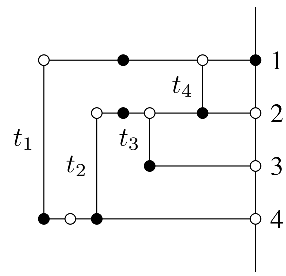


Fig. 1: A bridge network. All unlabeled edges have weight 1.

To convey the flavor of this result, we briefly sketch an example; the details will appear later. Take $k = 2$ and $n = 4$. Let $u = 2143$ and $w = 4321$. The Richardson variety $\mathring{X}_u^w$ projects birationally to the positroid $\mathring{\Pi}_{u,w}$. Fix the reduced word $\mathbf{w} = s_1 s_2 \boldsymbol{s_3} s_2 \boldsymbol{s_1} s_2$ for w. The PDS $\mathbf{u}$ for u in $\mathbf{w}$ comprises the bolded entries in $\mathbf{w}$, so we have Deodhar parametrization

$$(t_1,t_2,t_3,t_4) \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x_2(t_1)\dot{s}_1^{-1}x_2(t_2)\dot{s}_3^{-1}x_2(t_3)x_1(t_4) = \begin{bmatrix} 0 & 1 & t_3 & t_2 \\ -1 & -t_4 & 0 & t_1 \end{bmatrix} \tag{1.1}$$

We claim that we can obtain the same map from a bridge graph. Indeed, The bridge graph in Figure 1 yields the parametrization:

$$(t_1,t_2,t_3,t_4) \mapsto \begin{bmatrix} 1 & t_4 & 0 & -t_1 \\ 0 & 1 & t_3 & t_2 \end{bmatrix} \tag{1.2}$$

Note that the two parametrizations send the point (t_1, t_2, t_3, t_4) to representative matrices which have the same row space, and hence correspond to the same point in $\mathrm{Gr}(2, 4)$.

This work builds on a number of earlier results. Postnikov's Le-diagrams, which index positroid varieties, provided the first link between planar networks and PDS's. A Le-diagram is a Young diagram filled with 0's and $+$'s according to certain rules. There is a beautiful bijection between Le-diagrams and PDS's of *Grassmannian* permutations, permutations with a single descent at position k. Moreover, Postnikov constructed a planar network from each Le-diagram, which yields a parametrization of the corresponding positroid variety (Postnikov, 2006).

Talaska and Williams explored the link between distinguished subexpressions and network parametrizations further in (Talaska and Williams, 2013). They considered Deodhar components of $\mathcal{F}\ell(n)$ indexed by all distinguished subexpressions of Grassmannian permutations, not just PDS's. These Deodhar components project isomorphically to subsets of $\mathrm{Gr}(k, n)$, and the projections give a decomposition of $\mathrm{Gr}(k, n)$. Marsh and Rietsch's work yields a unique parametrization for each component (Marsh and Rietsch, 2004).

Talaska and Williams proved that each of these parametrizations arises from a network, which they constructed explicitly. For components indexed by PDS's, they recovered the planar networks corresponding to Postnikov's Le diagrams; for the remaining components, their networks are nonplanar.

Here, we relax the requirement that our Deodhar components be indexed by subexpressions of Grassmannian permutations. Instead, we restrict our attention to Deodhar components which correspond to PDS's, and which project isomorphically to a subset of $\mathrm{Gr}(k, n)$. There is a family of such components for each positroid variety, which in turn gives a family of parametrizations. We construct a planar network for each such parametrization. The Le-diagrams defined by Postnikov, and recovered by Talaska and Williams, are a special case of this result. This is an extended abstract for a longer paper, which may be found at arXiv:1411.2997 [math.CO].

2 Background

2.1 Notation

Let S_n denote the symmetric group in n letters, with simple generators $s_1, \ldots, s_{n-1}$, and let ℓ denote the standard length function on S_n. For $u, w \in S_n$, we write $u \leq w$ to denote a relation in the (strong) Bruhat order. A factorization $u = vw \in S_n$ is *length additive* if $\ell(u) = \ell(v) + \ell(w)$. We write $[a]$ for the set of integers $\{1, \ldots, a\}$. If $a < b$, let $[a, b] = \{a, a+1, \ldots, b\}$. Otherwise, let $[a, b] = \emptyset$. Let $w([a])$ be the set $\{w(1), \ldots, w(a)\}$. By $S_k \times S_{n-k}$, we mean the subgroup of S_n which fixes the sets $[k]$ and $[k+1, n]$. We denote the set of all k-elements subsets of n by $\binom{[n]}{k}$.

All functions and permutations act on the left. For $a \leq b$, we let $x_{(a,b)}(t)$ denote the *elementary matrix* with 1's along the main diagonal, a single nonzero entry t in position (a, b), and 0's everywhere else. Let $x_i(t)$ denote $x_{(i,i+1)}(t)$.

2.2 Bruhat intervals and bounded affine permutations

Fix $k \leq n$. The k-Bruhat order on S_n, introduced by Bergeron and Sottile in (Bergeron and Sottile, 1998), is defined as follows. For $u, w \in S_n$, we say that w is a k-cover of u, written $w \gtrdot_k u$, if $w \gtrdot u$ in Bruhat order and $w([k]) \neq u([k])$. To obtain the k-Bruhat order on S_n, we take the transitive closure of these cover relations. We use $\leq_k$ to denote k-Bruhat order, and write $[u, w]_k$ for the interval $\{v \mid u \leq_k v \leq_k w\}$.

Following (Knutson et al., 2013, Section 2.3), we define an equivalence relation on k-Bruhat intervals by setting $[u, w]_k \sim [x, y]_k$ if there is some $z \in S_k \times S_{n-k}$ such that $x = uz$ and $y = wz$, with both factorizations length additive. We write $\langle u, w \rangle_k$ for the equivalence class of $[u, w]_k$, and denote the set of all such classes by $\mathcal{Q}(k, n)$. There is a partial order on $\mathcal{Q}(k, n)$, defined by setting $\langle u, w \rangle_k \leq \langle x, y \rangle_k$ if there exist representatives $[u', w']_k$ of $\langle u, w \rangle_k$ and $[x', y']_k$ of $\langle x, y \rangle_k$ with $[x', y']_k \subseteq [u', w']_k$. The poset $\mathcal{Q}(k, n)$ is anti-isomorphic to a special case of the posets that appeared in (Brown et al., 2006) and (Rietsch, 2006).

Definition 2.1. *A **bounded affine permutation** of type (k, n) is a bijection $f : \mathbb{Z} \to \mathbb{Z}$ which satisfies the following criteria:*

1. *$f(i + n) = f(i) + n$ for all $i \in \mathbb{Z}$.*
2. *$i \leq f(i) \leq i + n$ for all $i \in \mathbb{Z}$.*
3. *$\frac{1}{n} \sum_{i=1}^{k} (f(i) - i) = k$*

We write $\mathrm{Bound}(k,n)$ for the set of bounded affine permutations of type (k,n). Let $t_{(i,j)} : \mathbb{Z} \to \mathbb{Z}$ be the map which interchanges $i + rn$ and $j + rn$ for all $r \in \mathbb{Z}$. The ***Bruhat ordering*** on $\mathrm{Bound}(k,n)$ is the transitive closure of the relations given by $f \to g$ if $f, g \in \mathrm{Bound}(k,n)$ and there exists $i < j \in \mathbb{Z}$ such that $f(i) < f(j)$ and $g = ft_{(i,j)}$. The poset $\mathrm{Bound}(k,n)$ is anti-isomorphic to Postnikov's poset of *decorated permutations*. For further discussion, see (Knutson et al., 2013).

For $J \in \binom{[n]}{k}$, define $t_J \in \mathrm{Bound}(k,n)$ by setting

$$i \mapsto \begin{cases} i+n & i \in J \\ i & i \in [n]\backslash J \end{cases} \tag{2.1}$$

and extending periodically. Let $\langle u, w\rangle_k \in \mathcal{Q}(k,n)$. The function

$$f_{u,w} = ut_{[k]}w^{-1} \tag{2.2}$$

lies in $\mathrm{Bound}(k,n)$. The map $\langle u, w\rangle_k \mapsto f_{u,w}$ gives a well-defined isomorphism of posets from $\mathcal{Q}(k,n)$ to $\mathrm{Bound}(k,n)$ (Knutson et al., 2013, Section 3.4). Essentially the same isomorphism appears in (Williams, 2007).

2.3 Grassmannians, flag varieties, and Richardson varieties

Let $\mathrm{Gr}(k,n)$ denote the Grassmannian of k-dimensional linear subspaces of the vector space $\mathbb{C}^n$. We realize $\mathrm{Gr}(k,n)$ as the space of full-rank $k \times n$ matrices modulo row operations; a matrix M represents the space spanned by its rows. We number the rows of our matrices from top to bottom, and the columns from left to right.

The *Plücker embedding*, denoted p, maps $\mathrm{Gr}(k,n)$ into the projective space $\mathbb{P}^{\binom{n}{k}-1}(\mathbb{C})$ with homogeneous coordinates x_J indexed by elements of $\binom{[n]}{k}$. For $J \in \binom{[n]}{k}$, and M a $k \times n$ matrix, let $\Delta_J(M)$ denote the minor of M with columns indexed by J. Let V be a k-dimensional subspace of $\mathbb{C}^n$ with representative matrix M. Then $p(V)$ is the point defined by $x_J = \Delta_J(M)$. The homogeneous coordinates Δ_J are known as *Plücker coordinates* on $\mathrm{Gr}(k,n)$. The *totally nonnegative Grassmannian* $\mathrm{Gr}_{\geq 0}(k,n)$ is the subset of $\mathrm{Gr}(k,n)$ whose Plücker coordinates are all nonnegative real numbers, up to multiplication by a common scalar.

The *flag variety* $\mathcal{F}\ell(n)$ is an algebraic variety whose points correspond to flags

$$V_\bullet = \{0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n\} \tag{2.3}$$

where V_i is a linear subspace of $\mathbb{C}^n$ of dimension i. We realize $\mathcal{F}\ell(n)$ as the quotient of $\mathrm{GL}(n)$ by the left action of B, the group of $n \times n$ lower triangular matrices. The projection $\pi_k : \mathcal{F}\ell(n) \to \mathrm{Gr}(k,n)$ carries a flag $V_\bullet$ to the k-plane V_k.

We recall the definitions of Schubert and Richardson varieties, following the conventions of (Knutson et al., 2013, Section 4). For a subset J of $[n]$, let $\mathrm{Project}_J : \mathbb{C}^n \to \mathbb{C}^J$ be projection onto the coordinates indexed by J. For a permutation $w \in S_n$, we define the *Schubert cell* corresponding to w by

$$\mathring{X}_w = \{V_\bullet \in \mathcal{F}\ell(n) \mid \dim(\mathrm{Project}_{[j]}(V_i)) = |w([i]) \cap [j]| \text{ for all } i\} \tag{2.4}$$

Similarly, we define the *opposite Schubert cell* by

$$\mathring{X}^w = \{V_\bullet \in \mathcal{F}\ell(n) \mid \dim(\mathrm{Project}_{[n-j+1,n]}(V_i)) = |w([i]) \cap [n-j+1,n]| \text{ for all } i\} \tag{2.5}$$

We define the *open Richardson variety* $\mathring{X}_u^w$ to be the transverse intersection $\mathring{X}_u \cap \mathring{X}^w$. The variety $\mathring{X}_u^w$ is empty unless $u \leq w$, in which case it has dimension $\ell(w) - \ell(u)$ (Kazhdan and Lusztig, 1980). Open Richardson varieties form a stratification of $\mathcal{F}\ell(n)$ which refines the Schubert stratification.

2.4 Positroid varieties

Let $V \in \mathrm{Gr}_{\geq 0}(k,n)$. The indices of the non-vanishing Plücker coordinates of V give a set $\mathcal{J} \subseteq \binom{[n]}{k}$ called the *matroid* of V. We define the *matroid cell* $\mathcal{M}_{\mathcal{J}}$ as the locus of points $V \in \mathrm{Gr}_{\geq 0}(k,n)$ with matroid $\mathcal{J}$. The nonempty matroid cells in $\mathrm{Gr}_{\geq 0}(k,n)$ are the *positroid cells* defined by Postnikov. Positroid cells form a stratification of $\mathrm{Gr}_{\geq 0}(k,n)$, and each cell is homeomorphic to $(\mathbb{R}^+)^d$ for some d (Postnikov, 2006, Theorem 3.5).

The positroid stratification of $\mathrm{Gr}_{\geq 0}(k,n)$ extends to the complex Grassmannian $\mathrm{Gr}(k,n)$. Taking the Zariski closure of a positroid cell of $\mathrm{Gr}_{\geq 0}(k,n)$ in $\mathrm{Gr}(k,n)$ gives a *closed positroid variety*. For a closed positroid variety $\Pi \subseteq \mathrm{Gr}(k,n)$, we define the *open positroid variety* $\mathring{\Pi} \subset \Pi$ by taking the complement in Π of all lower-dimensional positroid varieties. These open positroid varieties give a stratification of $\mathrm{Gr}(k,n)$ (Rietsch, 1999; Knutson et al., 2013). Positroid varieties in $\mathrm{Gr}(k,n)$ may be defined in numerous other ways. For instance, there is is a beautiful description of positroid varieties as intersections of cyclically permuted Schubert varieties (Postnikov, 2006; Knutson et al., 2013).

Positroid varieties in $\mathrm{Gr}(k,n)$ coincide with projected Richard varieties (Knutson et al., 2013, Section 5.4). Indeed, let $u \leq_k w$. The projection π_k maps $\mathring{X}_u^w$ isomorphically onto its image, which is an open positroid variety $\mathring{\Pi}_{u,w}$. If $\langle u', w' \rangle_k = \langle u, w \rangle_k$ then $\mathring{\Pi}_{u,w} = \mathring{\Pi}_{u',w'}$, so each class in $\langle u, w \rangle_k \in \mathcal{Q}(k,n)$ corresponds to a unique positroid variety. This correspondence gives a poset isomorphism between $\mathcal{Q}(k,n)$ and the poset of positroid varieties, ordered by *reverse* inclusion (Knutson et al., 2013, Section 5.4).

Since $\mathcal{Q}(k,n)$ is isomorphic to the poset of positroid varieties, so is $\mathrm{Bound}(k,n)$. We write Π_f for the positroid variety corresponding to a bounded affine permutation f. There are numerous other combinatorial objects that index positroid varieties; see (Postnikov, 2006; Knutson et al., 2013).

2.5 Plabic graphs and bridge graphs

A *plabic graph* is a planar graph embedded in a disk, with each vertex colored black or white. (Plabic is short for "planar bicolored.") The boundary vertices are numbered $1, 2, \ldots, n$ in clockwise order, and all boundary vertices have degree one. We call the edges adjacent to boundary vertices *legs* of the graph, and a leaf adjacent to a boundary vertex a *lollipop*.

Postnikov introduced plabic graphs in (Postnikov, 2006, Section 11.5), where he used them to construct parametrizations of positroid cells in the totally nonnegative Grassmannian. In this paper, we follow the more restrictive conventions of (Postnikov et al., 2009). In particular, we require our plabic graphs to be bipartite, with the black and white vertices forming the partite sets. Further, we restrict our attention to *reduced, perfectly orientable* plabic graphs, which satisfy some additional combinatorial conditions. See (Postnikov, 2006, Section 12).

Postnikov associates a *decorated permutation* σ_G to each plabic graph G with n boundary vertices; that is, a permutation with each fixed point colored black or white. If G is a reduced graph, each fixed point of σ_G corresponds to a boundary leaf (Postnikov, 2006). Let

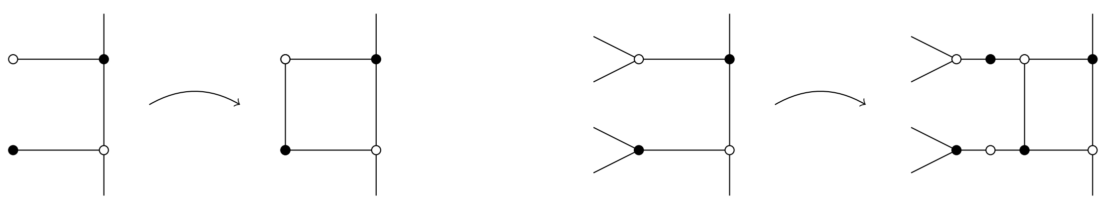
$$k = |\{a \in [n] \mid \sigma_G(a) < a \text{ or } \sigma_G(a) = a \text{ and } G \text{ has a white boundary leaf at } a\}| \tag{2.6}$$

Then there is a unique $f_G \in \mathrm{Bound}(k,n)$ corresponding to G such that $f_G(i) \equiv \sigma_G(i) \pmod n$ (Knutson et al., 2013). Thus we have a correspondence between plabic graphs and positroid varieties, with $\Pi_G = \Pi_{f_G}$, which assigns a family of reduced plabic graphs to each positroid variety.

We describe a way to build plabic graphs inductively by adding new edges, called *bridges*. The resulting graphs are called *bridge graphs*. Versions of this construction appear in (Arkani-Hamed et al., 2012) and (Lam, 2013b).

Begin with a plabic graph G. To add a bridge, we choose a pair of boundary vertices $a < b$ such every $c \in [a+1, b-1]$ is a lollipop, and $f_G(a) > f_G(b)$. Our new edge has one vertex on the leg at a, and one on the leg at b. If a (respectively b) is a lollipop, then the leaf at a must be white (black), and we use that boundary leaf as one endpoint of the bridge. If a (respectively b) is not a lollipop, we instead insert a white (black) vertex in the middle of the leg at a (respectively, b). After adding the bridge, we insert additional vertices of degree two or change the color of boundary vertices as needed to obtain a bipartite graph. (See Figure 2.) The following is straightforward.

Proposition 2.2. *Suppose G is reduced. Choose $1 \leq a < b \leq n$ such that $f_G(a) > f_G(b)$, and each $c \in [a+1, b-1]$ is a lollipop. Let G' be the graph obtained by adding an (a,b)-bridge to G. Then G' is a reduced plabic graph, and $f'_G = f \circ (a,b) \in \mathrm{Bound}(k,n)$.*



(a) Adding a bridge between boundary leaves.

(b) Adding a bridge between legs which are not boundary leaves. Note that after adding the bridge, we add additional vertices of degree 2 to create a bipartite graph.

Fig. 2: Adding bridges to a plabic graph.

The zero-dimensional positroid varieties correspond to the points in $\mathrm{Gr}(k,n)$ which have a single non-zero Plücker coordinate J. There is a unique reduced plabic graph for each $J \in \binom{[n]}{k}$, which has n lollipops. The k lollipops corresponding to elements of J are white; the rest are black. We call a plabic graph consisting only of lollipops a *lollipop graph*.

Let $u \leq_k w$. Then by (2.2), together with the fact that $ut_{[k]} = t_{u([k])}u$, we have $f_{u,w} = t_{u([k])}uw^{-1}$ To construct a *bridge graph* for $\Pi_{\langle u,w \rangle_k}$ we begin with the lollipop graph corresponding to $u([k])$, and successively add bridges, making sure the hypotheses of Proposition 2.2 are satisfied at each step, until we have a graph with bounded affine permutation $f_{u,w}$.

2.6 Parametrizations from plabic graphs

Let G be a reduced plabic network with e edges, and assign weights $t_1, \ldots, t_e$ to the edges of G. Postnikov defined a surjective map from the space of positive real edge weights of G to the positroid cell $\left(\mathring{\Pi}_G\right)_{\geq 0}$ in $\mathrm{Gr}_{\geq 0}(k,n)$, called the *boundary measurement map* (Postnikov, 2006, Section 11.5). Postnikov, Speyer and Williams re-cast this construction in terms of certain matchings (Postnikov et al., 2009, Section 4-5), an approach Lam developed further in (Lam, 2013b). Muller and Speyer showed that we can extend the boundary measurement map to the space of nonzero complex edge weights, and obtain a regular map; specializing all but an appropriate set of edges to 1 gives a birational map to a dense subset of $\mathring{\Pi}_G$ (Muller and Speyer, 2014). If G is a bridge graph, we have a simple procedure for constructing such a parametrization.

Note that for a bridge graph G, we have $\Pi_G = \Pi_{u,w}$ for some $u \leq_k w$. Let $d = \dim(\Pi_G)$. Assign weights $t_1, \ldots, t_d$ to each bridge, in the order the bridges were added, and set all other edge weights to 1. Begin with a $k \times n$ matrix which has a single nonzero Plücker coordinate indexed by the columns $u([k])$, and whose remaining columns consist of all 0's. Say the r^{th} bridge is from a to b, with $a < b$, and let

$$\theta = |u([k]) \cap [a+1, b-1]|$$

When we add the r^{th} bridge to the graph, we multiply our matrix on the right by $x_{(a,b)}((-1)^\theta t_r)$

2.6.1 Example

Let $n = 4$, $k = 2$. Let $w = 4321$ and $u = 2143$. Then $u \leq_2 w$, and $\langle u, w\rangle_2$ corresponds to the big cell in $\mathrm{Gr}(2,4)$. We have $u([2]) = \{1, 2\}$. The bounded affine permutation corresponding to $\langle u, w\rangle_2$ is given by $f_{u,w} = 3456$. Starting with the plabic graph corresponding to $t_{u([2])} = 5634$ we successively add bridges

$$(1,4) \to (2,4) \to (2,3) \to (1,2). \tag{2.7}$$

The corresponding bridge graph is shown in Figure 1.

We next construct the corresponding matrix parametrization. Since $u([2]) = \{1, 2\}$, we start with the $k \times n$ matrix which has a copy of the identity in its first two columns, and multiply on the right by the factors $x_{(a,b)}(\pm t_r)$ as in (2.6). So we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -t_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & t_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & t_3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & t_4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & t_4 & 0 & -t_1 \\ 0 & 1 & t_3 & t_2 \end{bmatrix} \tag{2.8}$$

2.7 Positive distinguished subexpressions

Let $\mathbf{w} = s_{i_1} \cdots s_{i_m}$ be a reduced word for $w \in S_n$. A *subexpression* $\mathbf{u}$ of $\mathbf{w}$ is obtained by replacing some of the factors s_{i_j} of $\mathbf{w}$ with the identity permutation, which we denote by 1. Following (Marsh and Rietsch, 2004), we write $\mathbf{u} \preceq \mathbf{w}$ to indicate that $\mathbf{u}$ is a subexpression of $\mathbf{w}$. We denote the t^{th} factor of $\mathbf{u}$, which may be either 1 or a simple transposition, by u_t, and write $u_{(t)}$ for the product $u_1 u_2 \ldots u_t$. Set $u_0 = u_{(0)} = 1$.

Given a subexpression $\mathbf{u} \preceq \mathbf{w}$, we say $\mathbf{u}$ is *a subexpression for* $u = u_1 u_2 \cdots u_r$.

Definition 2.3. *A subexpression* $\mathbf{u}$ *of* $\mathbf{w}$ *is called* ***positive distinguished*** *if for all* $1 \leq j \leq m$, *we have*

$$u_{(j-1)} < u_{(j-1)} s_{i_j}. \tag{2.9}$$

We will abbreviate the phrase "positive distinguished subexpression" to PDS.

2.8 Wiring diagrams, subexpressions, and bridge diagrams

Wiring diagrams represent words in S_n visually. Let w in S_n, and fix a word $\mathbf{w}$ of w. For an example, see Figure 3b. Each crossing between the wires represents a transposition s_i, where $i-1$ is the number of wires in the diagram which pass directly above the crossing. However, the crossings in the diagram for $\mathbf{w}$ appear in the opposite order as the corresponding generators in the word $\mathbf{w}$. If $w(s) = t$, the wire with left endpoint s has right endpoint t.

We now define *bridge diagrams*, which represent subexpressions of reduced words. To draw the *bridge diagram* for a subexpression $\mathbf{u} \preceq \mathbf{w}$ we take a wiring diagram for $\mathbf{u}$, and draw dashed crosses between adjacent wires to represent the additional crossings from $\mathbf{w}$. We call these dashed crosses *bridges*, by analogy with bridge graphs. For an example, see Figure 3a. If $\mathbf{u} \preceq \mathbf{w}$ is a bridge diagram, we label wires in the underlying diagram for $\mathbf{u}$ by their right endpoints. A wire is *isolated* if there are no bridges on that wire.

2.9 Deodhar parametrizations of positroid varieties

We review Deodhar's decompositions of the flag variety, as well as Marsh and Rietsch's parametrizations of Deodhar components, and their projections to the Grassmannian (Deodhar, 1985; Marsh and Rietsch, 2004). Our discussion follows (Talaska and Williams, 2013, Section 4), but with slightly different conventions.

To construct a Deodhar decomposition of $\mathcal{F}\ell(n)$, we first fix a reduced word $\mathbf{w}$ for each $w \in S_n$. We have a Deodhar component $\mathcal{R}_{\mathbf{u},\mathbf{w}}$ for each distinguished subexpression $\mathbf{u} \preceq \mathbf{w}$ with

$$\mathring{X}_u^w = \bigsqcup_{\mathbf{u} \preceq \mathbf{w} \text{ distinguished}} \mathcal{R}_{\mathbf{u},\mathbf{w}}. \tag{2.10}$$

To define a Deodhar decomposition of the Grassmannian, fix a Deodhar decomposition of $\mathcal{F}\ell(n)$, and choose a representative $\langle u, w \rangle_k$ for each class in $\mathcal{Q}(k,n)$. For each of the selected $u \leq_k w$, and each distinguished subexpression $\mathbf{u} \preceq \mathbf{w}$, the *Deodhar component* of $\mathrm{Gr}(k,n)$ corresponding to $\mathbf{u} \preceq \mathbf{w}$ is given by $\mathcal{D}_{\mathbf{u},\mathbf{w}} := \pi_k(\mathcal{R}_{\mathbf{u},\mathbf{w}})$.

We give the explicit matrix parametrization for $\mathcal{D}_{\mathbf{u},\mathbf{w}}$ in the case where $\mathbf{u} \preceq \mathbf{w}$ is a PDS. We write $\dot{s}_i$ for the matrix obtained from the $n \times n$ identity by replacing the 2×2 block whose upper left corner is at position (i,i) with the block matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{2.11}$$

Let $\mathbf{w} = s_{i_1} \cdots s_{i_m}$. Then we define g_j for $1 \leq i \leq m$ by

$$g_j = \left\{ \begin{array}{ll} x_{i_j}(t_j) & \text{if } s_{i_j} \text{ is not in } \mathbf{u} \\ \dot{s_{i_j}}^{-1} & \text{if } s_{i_j} \text{ is in } \mathbf{u} \end{array} \right\} \tag{2.12}$$

where the t_j are parameters, and define the set

$$G_{\mathbf{u},\mathbf{w}} := \{ g_m g_{m-1} \dots g_1 \in \mathrm{GL}(n) \mid t_j \in \mathbb{C}^\times \}. \tag{2.13}$$

Let $d = \ell(w) - \ell(u)$. The obvious map $(\mathbb{C}^\times)^d \to G_{\mathbf{u},\mathbf{w}}$ gives a parametrization of $\mathcal{R}_{\mathbf{u},\mathbf{w}}$ (Marsh and Rietsch, 2004). Composing with π_k, we have a parametrization of $\mathcal{D}_{\mathbf{u},\mathbf{w}}$. Let $j_1, \dots, j_d$ be the indices j

such that $s_{i_j} \notin \mathbf{u}$. For $1 \leq r \leq d$, set $\bar{r} = d + 1 - r$ and define

$$a_{\bar{r}} = u_{(j_r - 1)}(i_{j_r}) \qquad b_{\bar{r}} = u_{(j_r - 1)}(i_{j_r} + 1) \qquad \theta_{\bar{r}} = |u([k]) \cap [a_{\bar{r}} + 1, b_{\bar{r}} - 1])| \tag{2.14}$$

$$\beta_r = x_{(a_r, b_r)}((-1)^{\theta_r} t_{j_{\bar{r}}}). \tag{2.15}$$

Using the properties of PDS's and the commutation relations between the matrices $\dot{s_i}$ and $x_{(a,b)}$, we may write each $G \in G_{\mathbf{u},\mathbf{w}}$ in the form

$$G = (\dot{u}_m^{-1} \dot{u}_2^{-1} \cdots \dot{u}_1^{-1})(\beta_1 \beta_2 \cdots \beta_d). \tag{2.16}$$

For convenience, we renumber our parameters so that $\beta_r = x_{(a_r, b_r)}((-1)^{\theta_r} t_r)$.

2.9.1 Example continued

As before, let $n = 4$, $k = 2$. Let $\mathbf{w} = s_1 s_2 \boldsymbol{s_3} s_2 \boldsymbol{s_1} s_2 \in S_4$, and let $u = 2143$. The PDS $\mathbf{u}$ for u in $\mathbf{w}$ is comprised of the generators in bold, so we have a parametrization of $\mathcal{R}_{\mathbf{u},\mathbf{w}}$ by

$$G_{\mathbf{u},\mathbf{w}} = x_2(t_1) \dot{s_1}^{-1} x_2(t_2) \dot{s_3}^{-1} x_2(t_3) x_1(t_4). \tag{2.17}$$

Rewriting this in the form (2.16) and projecting to $\mathrm{Gr}(2, 4)$, we obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \dot{s_1}^{-1} \dot{s_3}^{-1} x_{(1,4)}(-t_1) x_{(2,4)}(t_2) x_2(t_3) x_1(t_4) = \begin{bmatrix} 0 & 1 & t_3 & t_2 \\ -1 & -t_4 & 0 & t_1 \end{bmatrix} \tag{2.18}$$

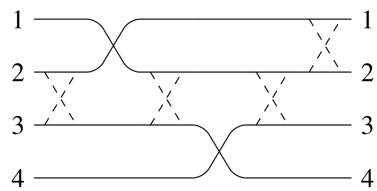
Note that this matrix has the same row span as the one obtained from the bridge graph above.

3 Sketch of the main proof

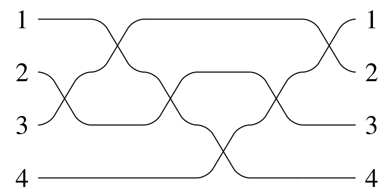
We sketch the proof of the forward direction of Theorem 1.1: every Deodhar parametrization corresponds to a bridge graph. The reverse direction also has a constructive proof, but the procedure is less visually elegant. Let $\mathbf{u} \preceq \mathbf{w}$ be a PDS with $u \leq_k w$. We construct a bridge graph G from the bridge diagram $\mathbf{u} \preceq \mathbf{w}$, as follows. For an example, see Figure 3.

Think of the right endpoints of the wires as boundary vertices of a bicolored graph embedded in a disk; the wires as paths with one endpoint on the boundary; and the new crossings as white-black bridges between these paths. Erase the tail of each wire from the left endpoint to the first bridge on that wire. If wire t is isolated, add a black lollipop at t if $u^{-1}(t) > k$, and a white lollipop if $u^{-1}(t) \leq k$. Finally, color the boundary vertices and add degree-two vertices as needed to obtain a bipartite graph. Remarkably, this process yields a planar embedding of a plabic graph whenever $\mathbf{u} \preceq \mathbf{w}$ is a PDS with $u \leq_k w$. Proving this requires a careful analysis of the possible configurations of bridge diagrams, using the properties of k-Bruhat order.

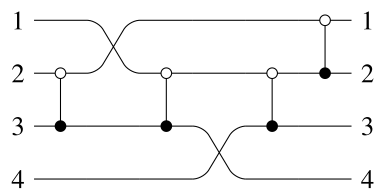
The parametrization arising from G is precisely the projected Deodhar parametrization corresponding to $\mathbf{u} \preceq \mathbf{w}$. Indeed, we construct the parametrization corresponding to G by taking a $k \times n$ matrix which has a single non-zero Plücker coordinate $u([k])$, and multiplying on the right by the matrices $\beta_1, \ldots, \beta_d$, where the β_r are defined as in (2.9). To see this, note that in the diagram, the r^{th} bridge from the left lies between wires a_r and b_r in the diagram for $\mathbf{u}$, where (a_r, b_r) is as in (2.14).



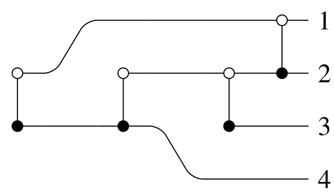
(a) The solid lines give the wiring diagram for $\mathbf{u}$. The dashed crosses represent bridges.



(b) Replacing each bridge at left with a crossing gives the wiring diagram for $\mathbf{w}$.



(c) To construct the bridge graph corresponding to $\mathbf{u} \preceq \mathbf{w}$, we first replace each bridge with a dimer, as shown above.



(d) Next, we delete the tail of each wire up to the first dimer on that wire. Adding degree-2 vertices as needed yields the desired plabic graph.

Fig. 3: Constructing a bridge graph from a bridge diagram. In this example, we have $k = 2$, $n = 4$, $\mathbf{w} = s_1 s_2 s_3 s_2 s_1 s_2$, and $\mathbf{u} = s_3 s_1$. Compare the figure at lower right with the bridge graph in Figure 1.

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