Formal Group Laws and Chromatic Symmetric Functions of Hypergraphs

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Abstract. If $f(x)$ is an invertible power series we may form the symmetric function

$$f(f^{-1}(x_1) + f^{-1}(x_2) + \cdots),$$

which is called a *formal group law*. We give a number of examples of power series $f(x)$ that are ordinary generating functions for combinatorial objects with a recursive structure, each of which is associated with a certain hypergraph. In each case, we show that the corresponding formal group law is the sum of the chromatic symmetric functions of these hypergraphs by finding a combinatorial interpretation for $f^{-1}(x)$. We conjecture that the chromatic symmetric functions arising in this way are Schur-positive.

Résumé. Si $f(x)$ est une série entière inversible, nous pouvons former la fonction symétrique

$$f(f^{-1}(x_1) + f^{-1}(x_2) + \cdots),$$

que nous appelons une *loi de groupe formel*. Nous donnons plusieurs exemples de séries entières $f(x)$ qui sont séries génératrices ordinaires pour des objets combinatoires avec une structure récursive, chacune desquelles est associée à un certain hypergraphe. Dans chaque cas, nous donnons une interprétation combinatoire à $f^{-1}(x)$, ce qui nous permet de montrer que la loi de groupe formel correspondante est la somme des fonctions symétriques chromatiques de ces hypergraphes. Nous conjecturons que les fonctions symétriques chromatiques apparaissant de cette manière sont Schur-positives.

Keywords: Formal group laws, hypergraph colorings, chromatic symmetric functions

1 Introduction

Formal group laws were first defined by S. Bochner in 1946 [2], who called them *formal Lie groups*. Specifically, a one-dimensional formal group law over a ring R is a formal power series $F(x, y) \in R[[x, y]]$ so that

- $F(x, 0) = x$, $F(0, y) = y$
- $F(F(x, y), z) = F(x, F(y, z))$.

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We will always take formal group laws to be one-dimensional and commutative, so that $F(x,y) = F(y,x)$, and we will take $R = \mathbb{Z}$. In this case, it is well known [9, IV.5] that formal group laws are exactly the formal power series of the form

$$F(x,y) = f(f^{-1}(x) + f^{-1}(y))$$

for some $f(x) \in \mathbb{Z}[[x]]$ with $f(0) = 0$ and $f'(0) = 1$, where $f^{-1}(x)$ is the compositional inverse of $f(x)$. More generally, we consider the power series F in infinitely many variables $x_1, x_2, \ldots$ given by

$$F(x_1, x_2, \ldots) = f(f^{-1}(x_1) + f^{-1}(x_2) + \cdots) \tag{1}$$

and from this point on we define formal group laws to be power series of the form (1).

The purpose of this paper is to develop a technique for showing that the formal group law F has nonnegative coefficients when $f(x)$ is an ordinary generating function for certain combinatorial objects. In each case we will use the language of hypergraphs colorings to give a combinatorial interpretation for the associated formal group law. A *hypergraph* (sometimes called a *set system*) is a pair $H = (V,E)$ where E is a family of subsets of V which we call the *edges* of H. We do not require that every edge e has the same number of elements, but we will always assume that V is finite and that no edge of H contains another edge. In the case that each edge of V has two elements, we say that H is an (ordinary) graph. We say that H is *connected* if V is not a disjoint union $V = V_1 \cup V_2$ of nonempty sets V_1, V_2 so that each edge $e \in E$ is contained in either V_1 or V_2. Every hypergraph H is the disjoint union of connected hypergraphs called the *connected components* of H.

We say that H is *linear* if $|e_1 \cap e_2| \leq 1$ for every $e_1, e_2 \in E$. If there is a labeling of V by the integers $[n] = \{1, 2, \ldots, n\}$ so that each $e \in E$ is an interval $I = \{a, a+1, \ldots, b-1, b\}$, we will say that H is an *interval hypergraph*. For example, if $V = [9]$ and E is the set of edges $\{1,2,3\}, \{3,4\}, \{5,6\}, \{6,7,8,9\}$ then $H = (V,E)$ is a linear interval hypergraph. All of the hypergraphs H appearing in our examples will be linear interval hypergraphs.

A *coloring* of H is a map $\chi : V \to \mathbb{P} = \{1, 2, 3, \ldots\}$. If e is an edge of H and χ is a coloring of H so that every element of e is the given the same color, we say that e is *monochromatic*, and we say that χ is *proper* if no edge $e \in E$ of H is monochromatic. (One might expect that we would define proper colorings so that each vertex in an edge must colored differently. But as Stanley notes in [12], in that case we could replace each edge by a complete ordinary graph and nothing new would be gained by considering hypergraphs.)

Let $x_1, x_2, \ldots$ be an infinite set of indeterminates. A *symmetric function* is a power series in $x_1, x_2, \ldots$ that is invariant under any permutation of the variables. For example,

$$p_n = x_1^n + x_2^n + \cdots$$

is a symmetric function known as the n-th *power sum* symmetric function.

The *chromatic symmetric function* X_H of a hypergraph H with vertex set V is the power series

$$X_H(x_1, x_2, \ldots) = \sum_{\chi} \prod_{v \in V} x_{\chi(v)}$$

with the sum taken over all proper colorings χ of H. Thus X_H is indeed a symmetric function, since permuting the colors of a proper coloring gives another proper coloring. The chromatic symmetric function X_G of an ordinary graph G was introduced by Stanley [11], who later generalized the notion to hypergraphs [12].

It will be useful in what follows to write the chromatic symmetric function X_H in terms of the power sum symmetric functions. The following theorem is a straightforward application of the inclusion-exclusion principle. A proof was for ordinary graphs given by Stanley [11, Theorem 2.5], who later extended it to hypergraphs [12, Theorem 3.2].

Theorem 1. *Let $H = (V, E)$ be a hypergraph. Then*

$$X_H = \sum_{S \subseteq E} (-1)^{|S|} p_{k_1} p_{k_2} \cdots p_{k_n}$$

where $k_1, \ldots, k_n$ are the sizes of the connected components of the hypergraph $H_S = (V, S)$ that has only the edges $e \in S$.

In Section 2, we will give a number of examples of power series $f(x)$ so that the associated formal group law F is a sum of chromatic symmetric functions, and hence has nonnegative coefficients. Specifically, we will describe sequences (A_n) of sets A_n of combinatorial objects T on the n nodes $1, 2, \ldots, n$ that have a recursive structure in the sense that one object may be inserted into any vertex of another object in a nice fashion, and certain "sub-objects" can be contracted to a point. Letting $a_n = |A_n|$, we form the generating function $f(x) = \sum_{n=1}^{\infty} a_n x^n$ and show that the corresponding formal group law has nonnegative coefficients. For example, we may let a_n be the sequence of Catalan numbers, Motzkin numbers, or the factorial numbers, corresponding to the sets A_n of binary trees, Motzkin paths and permutations, respectively. In each case, we will show that

$$f(f^{-1}(x_1) + f^{-1}(x_2) + \cdots) = \sum_{n=1}^{\infty} \sum_{T \in A_n} X_{H_T} \tag{2}$$

where $H_T = (V, E_T)$ is a particular hypergraph determined by the structure of the object $T \in A_n$, where $V = [n]$. The edges of T will be made up of the "sub-objects" of T that are minimal in the sense that they do not contain any other sub-objects except for singletons, and these edges will always be intervals in $[n]$. Furthermore, each H_T will be linear, so that H_T is a linear interval hypergraph.

While it is possible to give an axiomatic framework for proving equations of the form (2), we feel that for this extended abstract it is more useful to give individual examples. In each case, we will prove (2) by first giving a combinatorial interpretation for $f^{-1}(x)$. Letting C_n be the set of $T \in A_n$ so that the hypergraph H_T is connected, we will find that

$$f^{-1}(x) = \sum_{n=1}^{\infty} \sum_{T \in C_n} (-1)^{|E_T|} x^n. \tag{3}$$

Then we use (3) to prove (2) using the combinatorial interpretation of a composition of generating functions along with Theorem 1.

2 Examples

2.1 Lattice paths

Let L be a finite subset of $\mathbb{Z}$. Define a *L-admissible path* to be a map $P : [n] \to \mathbb{N}$ so that $P(1) = P(n) = 0$ and $P(i+1) - P(i) \in L$ for $i = 1, \ldots, n-1$. Let $A_{n,L}$ be the set of L-admissible paths $P : [n] \to \mathbb{N}$, let $a_{n,L} = |A_{n,L}|$ and define the generating function $f_L(x) = \sum_{n=1}^{\infty} a_{n,L} x^n$. If $L = \{-1, 1\}$ then an L-admissible path is called a *Dyck path*. Then $a_{n,L} = 0$ if n is even while $a_{2n+1} = C_n$, the n-th Catalan number, and

$$f_L(x) = \frac{1 - \sqrt{1 - 4x^2}}{2x}.$$

If $L = \{-1, 0, 1\}$ then L-admissible paths are called *Motzkin paths* and $a_{n,L} = M_{n+1}$ where M_n is the n-th Motzkin number, with

$$f_L(x) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x}.$$

There is an obvious way to insert one path into the vertex of another path. Formally, if P is an L-admissible path on $[n]$, $j \in [n]$, and P' is an L-admissible path on $[m]$, define $P(j \leftarrow P')$ to be the path on $[n + m - 1]$ given by

$$P(j \leftarrow P')(i) = \begin{cases} P(i) & \text{if } i < j \\ P'(i - j + 1) + P(j) & \text{if } j \leq i < j + m \\ P(i - m + 1) & \text{if } i \geq j + m. \end{cases}$$

It is clear that $P(j \leftarrow P')$ is an L-admissible path.

For any path $P : [n] \to \mathbb{N}$, define an *excursion* of P to be an interval $I = \{a, a + 1, \ldots, b\} \subseteq [n]$ so that $P(a) = P(b)$ and $P(i) \geq P(a)$ for $i \in I$. Thus restricting an L-admissible path P to an excursion I and translating produces another L-admissible path $Q \in A_{m,L}$ where $m = |I|$, so that excursions can be thought of as sub-objects of P. If $P \in A_{n,L}$ and $P' \in A_{m,l}$ then we see that $\{j, j + 1, \ldots, j + m - 1\}$ is an excursion of $P(j \leftarrow P')$.

We call a excursion I of P *minimal* if $|I| > 1$ and the only excursions of P properly contained in I are singletons. For each path P on $[n]$ we associate the hypergraph $H_P = (V, E_P)$ where $V = [n]$ and E_P is the set of minimal excursions of P. Fig. 1 gives an example of a path P and its associated hypergraph. Note that if I_1, I_2 are excursions of P with $I_1 \cap I_2 \neq \emptyset$, then $I_1 \cap I_2$ and $I_1 \cup I_2$ are also excursions. In particular, if I_1, I_2 are minimal excursions and $I_1 \cap I_2 \neq \emptyset$, then $I_1 \cap I_2$ must be a singleton. It follows that H_P is a linear interval hypergraph.

Now we can give a combinatorial interpretation of the formal group law associated with $f_L(x)$: it is the sum of the chromatic symmetric functions X_{H_P} for L-admissible paths P.

Theorem 2. *Let $L \subseteq \mathbb{Z}$ be finite. Then*

$$f_L(f_L^{-1}(x_1) + f_L^{-1}(x_2) + \cdots) = \sum_{n=1}^{\infty} \sum_{P \in A_{n,L}} X_{H_P}.$$

Note that in the case $L = \{0\}$, the only allowed path on $[n]$ is the constant path $P = 0$, so

$$f_L(x) = \frac{1}{1 - x} - 1, f_L^{-1}(x) = \frac{x}{1 + x}.$$

The minimal excursions of the constant path on $[n]$ are the intervals $\{i, i + 1\}$, and so

$$f_L(f_L^{-1}(x_1) + f_L^{-1}(x_2) + \cdots) = \frac{1}{1 - \frac{x_1}{1+x_1} - \frac{x_2}{1+x_2} - \cdots} - 1 = \sum_{n=1}^{\infty} X_{G_n} \tag{4}$$

where G_n is the ordinary graph with vertices $1, 2, \ldots, n$ and edges $\{i, i+1\}$ for $i = 1, \ldots, n-1$, sometimes called the n-vertex *path* (not to be confused with an L-admissible path P). A proper coloring of G_n is equivalent to a word $c_1 \cdots c_n$ with $c_i \in \mathbb{P}$ and $c_i \neq c_{i+1}$ for $i = 1, \ldots, n-1$. Such words are called *Smirnov* or *Carlitz* words, and if we assign a weight $x_{c_1} \cdots x_{c_n}$ to the word $c_1 \cdots c_n$ we see that (4) is the sum of the weights of all nonempty Smirnov words. This result can be found in Goulden and Jackson [5, 2.4.16], and was first found by MacMahon [7].

To prove Theorem 2, we will first find a combinatorial interpretation for $f^{-1}(x)$.

Lemma 3. *Let $L \subseteq \mathbb{Z}$ be finite and let $C_{n,L}$ be the set of L-admissible paths $P \in A_{n,L}$ so that H_P is a connected hypergraph. Let*

$$g_L(x) = \sum_{n=1}^{\infty} \sum_{P \in C_{n,L}} (-1)^{|E_P|} x^n.$$

Then $g_L(x) = f_L^{-1}(x)$.

Proof. Recall the usual combinatorial interpretation of the composition of ordinary generating functions. If $f(x) = \sum_{n=1}^{\infty} a_n x^n$ and $g(x) = \sum_{n=1}^{\infty} b_n x^n$ then

$$\begin{aligned} f(g(x)) &= \sum_{k=1}^{\infty} a_k \left(\sum_{n=1}^{\infty} b_n x^n \right)^k \\ &= \sum_{n=1}^{\infty} x^n \sum_{n_1 + \cdots + n_k = n} a_k b_{n_1} \cdots b_{n_k} \end{aligned} \tag{5}$$

where the inner sum is taken over all *compositions* of n, that is, k-tuples $(n_1, \ldots, n_k)$ of positive integers with $n_1 + \cdots + n_k = n$, with any positive number of parts k.

Now suppose that $a_n = |A_n|$ where A_n is the number of combinatorial structures of some type on n vertices. Then we interpret (5) as saying that the coefficient of x^n in $f(g(x))$ is given by taking a weighted sum indexed by partitions of $[n]$ into k disjoint subintervals for some k, where the summand $a_k b_{n_1} \cdots b_{n_k}$ represents the weight the set of all choices of a structure $T \in A_k$ to give the interval $[k]$ where each subinterval of size i in the partition is given the weight b_i.

Let $\Gamma_{n,L}$ be the set of tuples $(Q, Q_1, \ldots, Q_k)$ for some $k \geq 1$, where $Q \in A_{k,L}$ and each $Q_i \in C_{n_i,L}$ for some $n_i \in \mathbb{P}$, with $n_1 + \ldots + n_k = n$. Then the coefficient of x^n in $f_L(g_L(x))$ is

$$\sum_{(Q, Q_1, \ldots, Q_k) \in \Gamma_{n,L}} (-1)^{|E_{Q_1}| + \cdots + |E_{Q_k}|}. \tag{6}$$

We will show show that $f_L(g_L(x)) = x$ by showing that (6) is 1 if $n = 1$ and 0 otherwise.

Given $(Q, Q_1, \ldots, Q_k) \in \Gamma_{n,L}$ with $Q_i \in C_{n_i,L}$ we can form an L-admissible path $P \in A_{n,L}$ by inserting Q_1 into the first vertex of Q, inserting Q_2 into the second vertex of Q (which is the $k_1 + 1$-th vertex of $Q(1 \leftarrow Q_1)$), etc. That is, we let

$$P = Q(1 \leftarrow Q_1)(n_1 + 1 \leftarrow Q_2) \cdots (n_1 + \ldots + n_{k-1} + 1 \leftarrow Q_k).$$

Furthermore, we get a set $S \subseteq E_P$ by collecting all of the minimal edges of each Q_i as they appear within P, so that S consists of the translates $I + n_1 + \ldots + n_{i-1} := \{j + n_1 + \ldots + n_{i-1} : j \in I\}$ for the edges

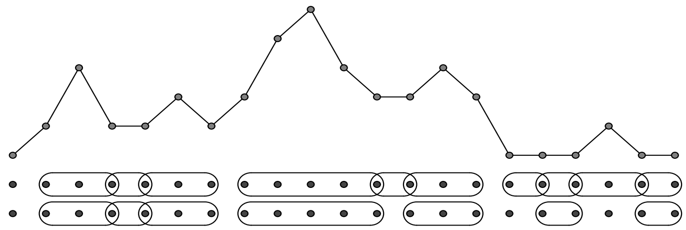

Fig. 1: An L-admissible path $P \in A_{n,L}$ with its minimal excursions $I \in E_P$ circled underneath it, where $L = \{-2, -1, 0, 1, 2\}$ and $n = 20$. Below that, the edges of an arbitrary subset $S \subseteq E_P$ are circled.

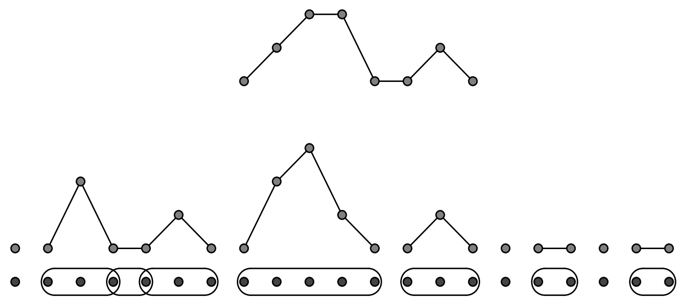

Fig. 2: A tuple $(Q, Q_1, \dots, Q_k) \in \Delta_{n,L}$, with $n = 20$ and $k = 8$, that is the result of applying the bijection ρ^{-1} to the pair $(P, S) \in \Delta_{n,L}$ from Fig. 1. Above is the path Q given by contracting each of the components of the hypergraph $H_S = ([n], S)$ to a point; below are the paths $Q_1, \dots, Q_k$ given by the individual components, with the minimal excursions $I \in E_{Q_i}$ circled.

$I \in E_{Q_i}$, for each i. This gives a map $\rho : \Gamma_{n,L} \to \Delta_{n,L}$ where $\Delta_{n,L}$ is the set of pairs (P, S) where $P \in A_{n,L}$ and $S \subseteq E_P$.

In fact, we can show that ρ is a bijection. Given an arbitrary $P \in A_{n,L}$ and $S \subseteq E_P$, let $I_1, I_2, \dots, I_k$ be the components of the hypergraph $H_S = ([n], S)$ with edge set $S \subseteq E_P$. Since the set of excursions is closed under non-disjoint unions, each I_i is an excursion of P, so restricting P to I_i and translating gives a path $Q_i \in A_{n_i,L}$ where $n_i = |I_i|$. The hypergraph H_{Q_i} is connected since I_i was a connected component of P, so $Q_i \in C_{n_i,L}$. Finally, we define Q by contracting each of the excursions I_i to a point. This defines the map ρ^{-1}; an example of its use is given in Fig. 2.

Applying the bijection ρ, (6) becomes

$$\sum_{P \in A_{n,L}} \sum_{S \subseteq E_P} (-1)^{|S|}.$$

The inner sum

$$\sum_{S \subseteq E_P} (-1)^{|S|} \tag{7}$$

is 0 unless E_P is empty, which only occurs if P is the path with one vertex, in which case (7) is 1. Thus $f_L(g_L(x)) = x$. □

Proof of Theorem 2. Again we use the combinatorial interpretation of a composition of ordinary generating functions. Let $f(x) = \sum_{n=1}^{\infty} a_n x^n$ and $g(x) = \sum_{n=1}^{\infty} b_n x^n$, and let $p_n = x_1^n + x_2^n + \cdots$ be the nth power sum symmetric function. Then

$$f(g(x_1)+g(x_2)+\cdots) = \sum_{k=1}^{\infty} a_k \left(\sum_{n=1}^{\infty} b_n x_1^n + \sum_{n=1}^{\infty} b_n x_2^n + \cdots\right)^k$$
$$= \sum_{k=1}^{\infty} a_k \left(\sum_{n=1}^{\infty} b_n p_n\right)^k$$
$$= \sum_{k=1}^{\infty} \sum_{n_1+\cdots+n_k=n} a_k b_{n_1}\cdots b_{n_k} p_{n_1}\cdots p_{n_k}. \tag{8}$$

In the case $f(x) = f_L(x)$ and $g(x) = f_L^{-1}(x)$, we can use Lemma 3 to rewrite (8) as

$$\sum_{n=1}^{\infty} \sum_{(Q,Q_1,\ldots,Q_k)\in\Gamma_{n,L}} (-1)^{|E_{Q_1}|+\cdots+|E_{Q_k}|} p_{n_1}\cdots p_{n_k} \tag{9}$$

where n_i is the number of vertices in the ith path Q_i. Applying the bijection ρ defined in the proof of 3, we see that (9) is

$$\sum_{n=1}^{\infty} \sum_{P\in A_n} \sum_{S\subseteq E_P} (-1)^{|S|} p_{n_1}\cdots p_{n_k} \tag{10}$$

where $n_1,\ldots,n_k$ are the sizes of the connected components of the hypergraph $H_S = (V,S)$. Using Theorem 1, (10) becomes

$$\sum_{n=1}^{\infty} \sum_{P\in A_n} X_{H_P}.$$

□

The fundamental tool in the proof of Theorem 2 is the ability to insert one path into any vertex of another path while preserving the edges, as well as the ability to contract edges (and connected unions of edges) to a point. This leads to the bijection ρ defined in the proof of 3, which allows us to use inclusion-exclusion to find a combinatorial interpretation for the compositional inverse $f_L^{-1}(x)$, which in turn allows us to find a combinatorial interpretation for the formal group law using the expansion of chromatic symmetric functions into power sums (Theorem 1). In what follows we will give more examples of this technique, but the details will be similar and the proofs will generally be omitted.

2.2 Plane trees with a fixed number of leaves

A *plane tree* (also called *embedded tree* is a rooted tree so that each node is equipped with an ordering of its children. A *leaf* of a tree T is a node that has no children. Let A_n be the set of plane trees with n leaves labeled $1, 2, \ldots, n$ left-to-right, with all other nodes unlabeled and with no node having exactly one child. (If nodes were allowed to have a single child, then each A_n would be infinite.)

There is an obvious way to insert one tree into any leaf of another that will allow us to replicate the proof of Theorem 2. If $T \in A_n$, i is a leaf of T, and $T' \in A_m$, we form the tree $T(i \leftarrow T')$ by replacing i

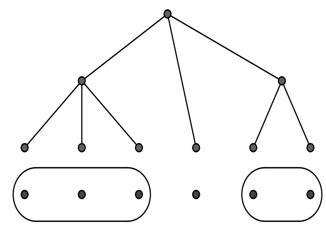

Fig. 3: A plane tree T with the edges $I \in E_T$ circled below.

with the root of T'. We let a *subtree* of T be a set of nodes T' so that T' is itself a tree that is a "down set", meaning that if v is a node of T' and u is descended from v in T, then $u \in T'$. Then let $H_T = (V, E_T)$ where E_T is the set of subsets I of leaves that form a complete set of siblings — all elements of I are children of a single node v, and every child of v is a leaf and is in I. Fig. 3 is an example of a tree T with its edge set E_T. Note that H_T has a very simple structure as a hypergraph, since the edges of H_T are always disjoint. In particular, H_T is a linear interval hypergraph.

Let $t_2, t_3, \ldots$ be indeterminates. Define the weight $w(T)$ of a plane tree to be $t_2^{k_2} t_3^{k_3} \cdots$ where k_i is the number of nodes in T that have exactly i children; thus if T is the tree in Fig. 3 then $w(T) = t_2 t_3^2$. Then let $f(x)$ be the generating function

$$f(x) = \sum_{n=1}^{\infty} x^n \sum_{T \in A_n} w(T).$$

If we set $t_2 = 1$ and $t_i = 0$ for $i > 2$ in the coefficient of x^n in $f(x)$ we get the number of binary trees with n leaves, the Catalan number C_{n-1}.

We have the following facts, whose proofs are similar to the proofs of Lemma 3 and Theorem 2.

Theorem 4 ([6, 8]). *Let C_n be the set of trees $T \in A_n$ so that the associated hypergraph H_T is connected. Then*

$$f^{-1}(x) = \sum_{n=1}^{\infty} x^n \sum_{T \in C_n} w(T)(-1)^{|E_T|}. \tag{11}$$

Furthermore,

$$f(f^{-1}(x_1) + f^{-1}(x_2) + \cdots) = \sum_{n=1}^{\infty} \sum_{T \in A_n} w(T) X_{H_T}. \tag{12}$$

Note that H_T is connected only when all the leaves of T are children of the root. In this case E_T consists of only one edge, which is the entire set of leaves. So C_n consists of this single tree T with weight $w(T) = t_n$ for each n, and by (11) we see

$$f^{-1}(x) = x - t_2 x^2 - t_3 x^3 - \cdots . \tag{13}$$

Equation (13) can also be seen as a corollary of a more general theorem due to Parker [8] giving a combinatorial interpretation of the compositional inverses of generating functions that count plane trees with certain restrictions. Equation (12) is due to Lenart [6, Theorem 3.2], although there it is given in a signed form.

We can also prove (12) directly, if we first define $f(x)$ so that $f^{-1}(x) = x - t_2x^2 - t_3x^3 - \cdots$ and let

$$F = \sum_{n=1}^{\infty} \sum_{T \in A_n} w(T) X_{H_T}.$$

Then we see that F obeys the functional equation

$$F = x_1 + x_2 + \cdots + t_2(F^2 - x_1^2 - x_2^2 - \cdots) + t_3(F^3 - x_1^3 - x_2^3 - \cdots) + \cdots \tag{14}$$

since a properly colored tree T consists either of a single vertex of any color, or a root whose descendants consist of some number $k \geq 2$ of subtrees $T_1, \ldots, T_k$ which are properly colored, where we do not allow the case where $T_1, \ldots, T_k$ are single vertices all colored the same. Then rearranging (12), we see

$$\begin{aligned} F - t_2F^2 - t_3F^3 - \cdots &= \left(x_1 - t_2x_1^2 - t_3x_1^3 - \cdots\right) + \left(x_2 - t_2x_2^2 - t_2x_2^2 - \cdots\right) + \cdots \\ f^{-1}(F) &= f^{-1}(x_1) + f^{-1}(x_2) + \cdots \end{aligned}$$

as desired.

2.3 Permutations

Let S_n be the set of permutations $\sigma : [n] \to [n]$, and let

$$f(x) = \sum_{n=1}^{\infty} |S_n| x^n = \sum_{n=1}^{\infty} n! x^n.$$

The power series $f(x)$ is nowhere convergent, but as a formal power series it still has a well-defined inverse $f^{-1}(x)$. As in our previous examples, there is a simple way to insert one permutation into another. In this case, it is most easily described in terms of permutation matrices. Let M_σ be the permutation matrix of $\sigma \in S_n$, given by the entries $a_{ij} = 1$ if $\sigma(j) = i$, with $a_{ij} = 0$ otherwise. Then if $\sigma \in S_m$, $j \in [m]$, and $\sigma' \in S_k$, we define the permutation $\sigma(j \leftarrow \sigma')$ by letting $M_{\sigma(j \leftarrow \sigma')}$ be the matrix given by inserting $M_{\sigma'}$ as a $k \times k$ block into the entry $(j, \sigma(j))$ of M_σ. For example, if $\sigma = 41523$, $j = 4$, and $\sigma' = 213$ then

$$M_\sigma = \begin{bmatrix} 0&1&0&0&0 \\ 0&0&0&\boxed{1}&0 \\ 0&0&0&0&1 \\ 1&0&0&0&0 \\ 0&0&1&0&0 \end{bmatrix}, \; M_{\sigma'} = \begin{bmatrix} 0&1&0 \\ 1&0&0 \\ 0&0&1 \end{bmatrix}, \; M_{\sigma(4 \leftarrow \sigma')} = \left[\begin{array}{ccc|ccc|c} 0&1&0&0&0&0&0 \\ \hline 0&0&0&0&1&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0 \\ \hline 0&0&0&0&0&0&1 \\ 1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \end{array}\right]$$

and so $\sigma(4 \leftarrow \sigma') = 6173245$. From here we see how to define the associated hypergraph H_σ for a permutation $\sigma \in S_n$. We let $\hat{E}_\sigma$ be the set of intervals $I \subseteq [n]$ so that σ maps I to another interval

$J \subseteq [n]$; thus $\{(j, \sigma(j)) : j \in I\}$ is the set of entries with 1's in an $|I| \times |I|$ block within M_σ which is a permutation matrix in itself. Then we define E_σ to be the minimal non-singleton elements of $\hat{E}_\sigma$. For example, if $\sigma = 659421387$ then E_σ consists of the intervals $\{1, 2\}, \{4, 5, 6, 7\}, \{8, 9\}$ since these map to the intervals $\{5, 6\}, \{1, 2, 3, 4\}$, and $\{7, 8\}$ respectively and they are minimal with respect to this property.

We then have the following facts, whose proofs are similar to the previous examples.

Theorem 5. *Let C_n be the set of permutations σ of S_n so that H_σ is connected. We have*

$$f^{-1}(x) = \sum_{n=1}^{\infty} \sum_{\sigma \in C_n} (-1)^{|E_\sigma|} x^n \tag{15}$$

and

$$f(f^{-1}(x_1) + f^{-1}(x_2) + \cdots) = \sum_{n=1}^{\infty} \sum_{\sigma \in S_n} X_{H_\sigma}.$$

A permutation $\sigma \in S_n$ is called *simple* if $E_\sigma = \{[n]\}$, so that σ does not map any proper non-singleton subinterval of $[n]$ to another subinterval of $[n]$. For example, the permutations 12 and 24153 are simple, but 253641 is not simple because it maps the interval $\{2, 3, 4, 5\}$ to the interval $\{3, 4, 5, 6\}$. We can also state this in terms of the permutation matrix M_σ: if σ is simple then M_σ has no $k \times k$ block that is itself a permutation matrix, unless $k = 1$ or n.

Clearly if $\sigma \in S_n$ is simple then H_σ is connected and so $\sigma \in C_n$. In fact, we will show that all but two of the permutations in C_n are simple for $n > 2$. It is not hard to show that if I_1, I_2 are distinct elements of E_σ with $I_1 \cap I_2 \neq \emptyset$ then $|I_1| = |I_2| = 2$. It follows that if H_σ is a connected hypergraph that is not simple then σ is either the identity $123 \cdots n$ or the reverse permutation $n \cdots 321$. In the latter two cases we have $E_\sigma = \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}\}$, so $|E_\sigma| = n - 1$. Then using (15) we find that

$$f^{-1}(x) = x - 2x^2 + \sum_{n=3}^{\infty} \left(2(-1)^{n-1} - s_n\right) x^n \tag{16}$$

where s_n is the number of simple permutations in S_n, which is the sequence $1, 2, 0, 2, 6, 46, 338, \ldots$ (Sequence A059372 in [10].) Equation (16) was found by Albert and Atkinson [1].

2.4 Other examples

There are a number of other power series $f(x)$ for which

$$f(f^{-1}(x_1) + f^{-1}(x_2) + \cdots)$$

can be shown to have nonnegative coefficients using these methods. For example, instead of taking A_n to be plane trees with any number of nodes but with leaves labeled $1, 2, \ldots, n$, we can take plane trees with each of its nodes labeled $1, 2, \ldots, n$. Let

$$f(x) = \sum_{n=1}^{\infty} \sum_{T \in A_n} w(T) x^n$$

where $w(T) = t_1^{k_1} t_2^{k_2} \cdots$ where k_i is the number of nodes of T with exactly i children. It is shown within the proof of the Lagrange inversion formula in [13, Theorem 5.4.2] that

$$\frac{x}{f^{-1}(x)} = 1 + t_1 x + t_2 x^2 + \cdots \tag{17}$$

and (17) can also be used to define $f(x)$. It is possible to prove that the corresponding formal group law has nonnegative coefficients using the technique we have described.

For simplicity of exposition we have only discussed the case where $f(x)$ is an ordinary generating function, but it is also possible to extend the methods to exponential generating functions. For example, we may take $f(x)$ to be the exponential generating function for labeled trees or labeled graphs. However, the hypergraphs that arise in the exponential case are not necessarily linear interval hypergraphs.

3 A conjecture

This section will assume a basic knowledge of symmetric functions as found in, e.g., [13, Chapter 7], and in particular knowledge of the *Schur functions* s_λ which form an important basis of the ring of symmetric functions Λ in infinity many variables $x_1, x_2, \ldots$. If a symmetric function has positive coefficients in a basis $\{b_\lambda\}$ of Λ we will say that it is *b-positive*; we will say that a hypergraph H is b-positive if X_H is. Based on numerical evidence, we make the following conjecture.

Conjecture 6. *Linear interval hypergraphs are Schur-positive.*

In particular, this conjecture would imply that all of the formal group laws discussed in 2.1, 2.2, 2.3 are Schur-positive.

In some cases we can prove Schur-positivity directly. For example, let $f(x) = x/(1-x)$. Stanley has shown [13, Exercise 7.47(k)] that (4) can be rewritten

$$f(f^{-1}(x_1) + f^{-1}(x_2) + \cdots) = \sum_{n=1}^{\infty} X_{G_n} = \frac{\sum_{i=1}^{\infty} e_i}{1 - \sum_{i=1}^{\infty} (i-1) e_i} - 1 \tag{18}$$

where e_i is the ith elementary symmetric function. It follows that the formal group law F corresponding to $f(x) = x/(1-x)$ is e-positive and hence Schur-positive, and all the paths G_n are e-positive as well. A linear interval hypergraph that is actually an ordinary graph is a disjoint union of paths, and so must also be e-positive. The Schur positivity of a disjoint union of paths also follows from results of Gessel [4] and Gasharov [3], where a combinatorial interpretation of the coefficients of X_G in the Schur basis is given when G is the incomparability graph of a $(3+1)$-free poset.

If H is the hypergraph with vertex set $[n]$ whose only edge is the whole set $[n]$ then $X_H = p_1^n - p_n$ since the only colorings of H that are not proper are the ones that assign all of H to a single color. It is not hard to see that X_H is Schur-positive in this case, and it follows that any hypergraph with all edges disjoint is Schur-positive. Recall from Example 2.2 that if

$$f^{-1}(x) = x - t_2 x^2 - t_3 x^3 - \cdots$$

then

$$f(f^{-1}(x_1) + f^{-1}(x_2) + \cdots) = \sum_T w(T) X_{H_T}$$

with the sum taken over plane trees T where $w(T)$ is a monomial in $t_2, t_3, \ldots$ and H_T is a hypergraph with all of its edges disjoint. It follows we may set $t_2, t_3, \ldots$ to be any sequence of nonnegative real numbers and the resulting formal group law will be Schur-positive, giving the following.

Theorem 7. *If $f(x) \in \mathbb{R}[[x]]$ so that $f^{-1}(x) = x - t_2x^2 - t_3x^3 - \cdots$, with each $t_i \geq 0$, then the corresponding formal group law is Schur-positive.*

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